STATIC BONDI ENERGY IN THE TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY

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Abstract

We consider Bondi's radiating metric in the context of the teleparallel equivalent of general relativity (TEGR). This metric describes the asymptotic form of a radiating solution of Einstein's equations. The total gravitational energy for this solution can be calculated by means of pseudo-tensors in the static case. In the nonstatic case, Bondi defines the *mass aspect* $m(u)$, which describes the mass of an isolated system. In this paper we express Bondi's solution in asymptotically spherical 3+1 coordinates, not in radiation coordinates, and obtain Bondi's energy in the static limit by means of the expression for the gravitational energy in the framework of the TEGR. We can either obtain the total energy or the energy inside a large (but finite) portion of a three-dimensional spacelike hypersurface, whose boundary is far from the source.

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I. Introduction

The concept of energy in general relativity is considerably more intricate than in any other branch of physics. Any physical phenomena, except gravitation, is defined and described on a specific space-time, which is usually the flat space-time. For these phenomena the concept of energy can be intuitively conceived and mathematically realized. Generically, energy is an attribute of some physical system whose dynamics takes place on the space-time. Gravitation, however, acquires a distinct status because the dynamics of the gravitational field is the dynamics of the space-time itself. Consequently, the definition of the gravitational energy is not straightforward.

The several attempts at defining the gravitational energy (pseudotensors, quasi-local energy, actions and Hamiltonians with surface terms) all agree in predicting the *total* energy of asymptotically flat gravitational fields. Moreover, there seems to exist a predominant point of view according to which the gravitational energy is not localizable, i.e., that there does not exist a gravitational energy *density*. These are probably the only two features shared by the various approaches, which are mostly based on the metric tensor. However, the very concept of a black hole lends support to the idea that gravitational energy is localizable. There is no process by means of which the gravitational mass inside a black hole can be made to vanish.

A detailed analysis of the structure of the pseudotensors shows that the (covariant) gravitational energy-momentum tensor would have to be defined by means of the first derivative of the metric tensor. But it is well known that it is not possible to write down a non-trivial covariant expression involving the first derivative of the metric, which captures the energy content of the field. However, it is possible to write down such covariant expressions with tetrads, and Møller noticed this fact long ago[1, 2, 3].

The question of localizability of the gravitational energy can be discussed in the framework of the teleparallel equivalent of general relativity (TEGR)[3, 4, 5, 6], which is an alternative geometrical formulation of Einstein's general relativity. The action integral of the TEGR is constructed entirely out of the torsion tensor. The analysis of the canonical structure of the TEGR[7] indicates the existence of a perfectly well defined gravitational energy density. Such existence is possible in principle, because the TEGR is defined in terms of tetrad fields. The torsion tensor allows the construction of a total divergence that transforms as a scalar (energy) density. In the 3+1

formulation of the TEGR the integral form of the Hamiltonian constraint equation $C = 0$ can be written as an energy equation of the type[8]

$$C \;=\; H \,-\, E \;=\; 0\;,$$

where E is the gravitational energy defined by

$$E_g \;=\; \frac{1}{8\pi G}\int_V d^3x\,\partial_i(eT^i)\;, \tag{1}$$

where $e = det(e_{(k)i})$, $\{e_{(k)i}\}$ are triads restricted to a three-dimensional spacelike hypersurface Σ, and T^i is the trace of the torsion tensor: $T^i = g^{ik}T_k = g^{ik}\,e^{(l)j}\,T_{(l)jk}$, $T_{(l)jk} = \partial_j e_{(l)k} - \partial_k e_{(l)j}$. V is an arbitrary three-dimensional volume of integration and G is the gravitational constant. This expression is simple and powerful. It has been successfully applied to rotating black holes[9], de Sitter space[10] and conical space-times[11]. The definition of gravitational energy in the TEGR may not be intuitively clear, but it is supported by its mathematical simplicity and by the applications to the space-times listed above. The use of (1) requires only the construction of the triads $\{e_{(k)i}\}$ with the appropriate boundary conditions, and which transform under the *global* SO(3) group.

The torsion tensor that appears in the Hamiltonian formulation of the TEGR is related to the antisymmetric component of the connection $\Gamma^i_{jk} = e^{(m)i}\partial_j e_{(m)k}$, whose curvature tensor is identically vanishing. Such connection defines a space with teleparalellism, or absolute paralelism, or else *fernparallelismus*, according to Schouten[12].

In this paper we investigate the energy of asymptotically flat gravitational waves, described by Bondi's radiating metric[13]. Since the metric describes an isolated system the application of (1) is possible as it stands, provided we consider the metric in the 3+1 spherical coordinates (t, r, θ, ϕ) at spacelike infinity, for which $t = constant$ defines a spacelike hypersurface. We note that the use of cartesian (rectangular) coordinates in the asymptotic limit is necessary for the evaluation of pseudotensors out of this metric.

We recall that the Arnowitt-Deser-Misner (ADM) energy[14] is not suitable for the analysis of gravitational radiation because it gives the *total* energy of the space-time, both from the source and from the emitted radiation, whereas the Bondi energy evaluated at null infinity furnishes only the energy

of the source, from which it is possible to derive the well known formula for the loss of mass.

The relevance of the definition (1) resides precisely in the fact that we can evaluate it on a large but *finite* volume V of the three-dimensional spacelike hypersurface, thereby not including the emitted radiation outside V. In view of the field equations (which are not considered here), the energy inside V turns out to be a decreasing function of time.

It is important to remark at this point that Bondi energy has been calculated in several geometrical frameworks, by different aproaches[15, 16, 17, 18, 19]. A commom feature of these approaches is that they yield the *total* energy of the field. In contrast, we will consider finite volumes of spacelike surfaces and obtain the energy contained within large spherical surfaces of radius r_o up to the $\frac{1}{r_o}$ term.

In the next section we briefly describe the Lagrangian and Hamiltonian formulations of the TEGR. In section III we compare our energy expression with Møller's expression. We show that both expressions agree for the *total* gravitational energy, but inspite of similarities they disagree when applied to finite volumes of the three-dimensional space. In section IV we write Bondi's metric in (t, r, θ, ϕ) coordinates at infinity and proceed to carry out the construction of triads for the spacelike hypersurfaces Σ. There exists an infinit number of triads that lead to the metric restricted to the three-dimensional hypersurface. However, only two of them will be considered in detail. In section V we calculate both the total energy of the field and the energy contained within a large sphere of radius r_o. The total energy obtained by means of (1), in which case the integration is made over the whole Σ, agrees with the known result for the Bondi energy in the *static* case. We also obtain the expression for the energy contained within a surface of constant radius r_o in the asymptotic region where the metric coefficients may be determined.

Notation: spacetime indices $\mu, \nu, ...$ and local Lorentz indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a{}_\mu$ and the spin connection $\omega_{\mu ab}$ yield the usual definitions of the torsion and curvature tensors: $R^a{}_{b\mu\nu} = \partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - ...$, $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - ....$ The flat spacetime metric is fixed by $\eta_{(0)(0)} = -1$.

II. The TEGR

The Lagrangian density of the TEGR in empty spacetime is given by

$$L(e,\omega,\lambda) \;=\; -ke(\frac{1}{4}T^{abc}T_{abc} \,+\, \frac{1}{2}T^{abc}T_{bac} \,-\, T^aT_a) \,+\, e\lambda^{ab\mu\nu}R_{ab\mu\nu}(\omega)\;. \quad (2)$$

where $k = \frac{1}{16\pi G}$, G is the gravitational constant; $e = det(e^a\,_\mu)$, $\lambda^{ab\mu\nu}$ are Lagrange multipliers and T_a is the trace of the torsion tensor defined by $T_a = T^b\,_{ba}$. The tetrad field $e_{a\mu}$ and the spin connection $\omega_{\mu ab}$ are completely independent field variables. The latter is enforced to satisfy the condition of zero curvature. Therefore this Lagrangian formulation is in no way similar to the usual Palatini formulation, in which the spin connection is related to the tetrad field via field equations. Later on we will introduce the tensor Σ_{abc} defined by

$$\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a \;\equiv\; T^{abc}\Sigma_{abc}\;.$$

The equivalence of the TEGR with Einstein's general relativity is based on the identity

$$eR(e,\omega) \;=\; eR(e) \,+\, e(\frac{1}{4}T^{abc}T_{abc} \,+\, \frac{1}{2}T^{abc}T_{bac} \,-\, T^aT_a) \,-\, 2\partial_\mu(eT^\mu)\;, \quad (3)$$

which is obtained by just substituting the arbitrary spin connection $\omega_{\mu ab} = {}^o\omega_{\mu ab}(e) + K_{\mu ab}$ in the scalar curvature tensor $R(e,\omega)$ in the left hand side; ${}^o\omega_{\mu ab}(e)$ is the Levi-Civita connection and $K_{\mu ab} = \frac{1}{2}e_a\,^\lambda e_b\,^\nu(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$ is the contorsion tensor. The vanishing of $R^a\,_{b\mu\nu}(\omega)$, which is one of the field equations derived from (2), implies the equivalence of the scalar curvature $R(e)$, constructed out of $e^a\,_\mu$ only, and the quadratic combination of the torsion tensor. It also ensures that the field equation arising from the variation of L with respect to $e^a\,_\mu$ is strictly equivalent to Einstein's equations in tetrad form. Let $\frac{\delta L}{\delta e^{a\mu}} = 0$ denote the field equations satisfied by $e^{a\mu}$. It can be shown by explicit calculations that

$$\frac{\delta L}{\delta e^{a\mu}} \;=\; \frac{1}{2}e\{R_{a\mu}(e) \,-\, \frac{1}{2}e_{a\mu}R(e)\}\;. \quad (4)$$

We refer the reader to refs.[7, 8] for additional details.

Throughout this section we will be interested in asymptoticaly flat space-times. The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints (to be shown below) constitute a *first class* set[7]. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (2). We still make use of the residual time dependent gauge symmetry to fix the usual time gauge condition $e_{(k)}{}^0 = e_{(0)i} = 0$. Because of $\omega_{0ab} = 0$, H does not depend on P^{kab}, the momentum canonically conjugated to ω_{kab}. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to P^{kab} yields $\dot{\omega}_{kab} = 0$. Thus in view of $\omega_{0ab} = 0$, ω_{kab} drops out from our considerations. The above gauge fixing can be understood as the fixation of a reference frame.

As a consequence of the above gauge fixing the canonical action integral obtained from (2) becomes[8]

$$A_{TL} = \int d^4x \{\Pi^{(j)k}\dot{e}_{(j)k} - H\} , \tag{5}$$

$$H = NC + N^iC_i + \Sigma_{mn}\Pi^{mn} + \frac{1}{8\pi G}\partial_k(NeT^k) + \partial_k(\Pi^{jk}N_j) . \tag{6}$$

N and N^i are the lapse and shift functions, and $\Sigma_{mn} = -\Sigma_{nm}$ are Lagrange multipliers. The constraints are defined by

$$C = \partial_j(2keT^j) - ke\Sigma^{kij}T_{kij} - \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2) , \tag{7a}$$

$$C_k = -e_{(j)k}\partial_i\Pi^{(j)i} - \Pi^{(j)i}T_{(j)ik} , \tag{7b}$$

with $e = det(e_{(j)k})$ and $T^i = g^{ik}e^{(j)l}T_{(j)lk}$. We remark that (5) and (6) are invariant under global SO(3) and general coordinate transformations.

If we assume the asymptotic behaviour

$$e_{(j)k} \approx \eta_{jk} + \frac{1}{2}h_{jk}(\frac{1}{r}) \tag{8}$$

for $r \to \infty$, then in view of the relation

$$\frac{1}{8\pi G}\int d^3x \partial_j(eT^j) = \frac{1}{16\pi G}\int_S dS_k(\partial_i h_{ik} - \partial_k h_{ii}) \equiv E_{ADM} \tag{9}$$

where the surface integral is evaluated for $r \to \infty$, the integral form of the Hamiltonian constraint $C = 0$ may be rewritten as

$$\int d^3x \left\{ ke\Sigma^{kij}T_{kij} + \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2) \right\} \;=\; E_{ADM} \; . \tag{10}$$

The integration is over the whole three dimensional space. Given that $\partial_j(eT^j)$ is a scalar density, from (9) and (10) we define the gravitational energy density enclosed by a volume V of the space as

$$E \;=\; \frac{1}{8\pi G}\int_V d^3x \partial_j(eT^j) \; . \tag{1}$$

It must be noted that E depends only on the triads $e_{(k)i}$ restricted to a three-dimensional spacelike hypersurface; the inverse quantities $e^{(k)i}$ can be written in terms of $e_{(k)i}$. From the identity (4) we observe that the dynamics of the triads does not depend on $\omega_{\mu ab}$. Therefore E_g given above does not depend on the fixation of any gauge for $\omega_{\mu ab}$. We briefly remark that the reference space which defines the zero of energy has been discussed in ref.[9].

We make now the important assumption that general form of the canonical structure of the TEGR is the same for any class of space-times, irrespective of the peculiarities of the latter (for the de Sitter space[10], for example, there is an *additional* term in the Hamiltonian constraint C). Therefore we assert that the integral form of the Hamiltonian constraint equation can be written as $C = H - E = 0$ for *any* space-time, and that (1) represents the gravitational energy for arbitrary space-times with any topology.

Before closing this section, let us recall that Müller-Hoissen and Nitsch[20] and Kopczyński[21] have shown that in general the theory defined by (2) faces difficulties with respect to the Cauchy problem. They have shown that in general six components of the torsion tensor are not determined from the evolution of the initial data. On the other hand, the constraints of the theory constitute a first class set provided we fix the six quantities $\omega_{0ab} = 0$ *before varying the action*[7]. This condition is mandatory and does not merely represent one particular gauge fixing of the theory. Since the fixing of ω_{0ab} yields a well defined theory with first class constraints, we cannot assert that the field configurations of the latter are gauge equivalent to configurations whose time evolution is not precisely determined. The requirement of local SO(3,1) symmetry plus the addition of $\lambda^{ab\mu\nu}R_{ab\mu\nu}(\omega)$ in (2) has the ultimate effect of discarding the connection. Although we have no proof, we believe

that the two properties above (the failure of the Cauchy problem and the fixation of $\omega_{0ab} = 0$) are related to each other.

Constant rotations constitute a basic feature of the teleparallel geometry. According to Møller[2], in the framework of the abolute parallelism tetrad fields, together with the boundary conditions, uniquely determine a *tetrad lattice*, apart from an arbitrary *constant rotation of the tetrads in the lattice*.

III. Møller's energy expression

Møller carried out several investigations regarding the localizability of the gravitational energy. He faced difficulties in establishing a covariant expression using the metric tensor[2], and because of this he arrived at an expression through the use of tetrads[2, 3]. According to Møller, this latter expression still has a difficulty in that it is not invariant under *local* Lorentz transformations. It is very instructive to compare expression (1) with Møller's expression. For the sake of this comparison, we will put aside the difficulty regarding the noninvariance with respect to local Lorentz transformations.

Møller presents an expression for the energy-momentum of the gravitational field. However we will only consider the energy expression. Translating into our notation, Møller's energy reads

$$E \;=\; -\int d^3x\,\partial_\lambda U_0{}^{0\lambda}\;, \tag{11}$$

where the potential in the integrand is given by

$$U_\mu{}^{\nu\lambda} \;=\; \frac{1}{8\pi G} e\,[e^{a\nu}\nabla_\mu e_a{}^\lambda + (\delta^\nu_\mu e^{a\lambda} - \delta^\lambda_\mu e^{a\nu})\nabla_\sigma e_a{}^\sigma] \qquad (12)\;.$$

In contrast with the notation of the previous section, all geometrical quantities in equations (11-15) are *four-dimensional* quantities. In (12) ∇ represents the covariant derivative with respect to the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$.

Møller's energy can be first rewritten as

$$E \;=\; \frac{1}{8\pi G}\int d^3x\,\partial_i(e\,e^{ak}\nabla_k e_a{}^i)\;. \tag{13}$$

By means of the identity

$$\nabla_k e_{aj} = \partial_k e_{aj} - \Gamma^{\sigma}_{kj} e_{a\sigma} \equiv -{}^o\omega_k{}^b{}_a e_{bj} \;, \tag{14}$$

where ${}^o\omega_{\mu ab}$ is the Levi-Civita connection,

$$^o\omega_{\mu ab} = -\frac{1}{2} e^c{}_\mu (\Omega_{abc} - \Omega_{bac} - \Omega_{cab}) \;,$$

$$\Omega_{abc} = e_{a\nu}(e_b{}^\mu \partial_\mu e_c{}^\nu - e_c{}^\mu \partial_\mu e_b{}^\nu) \;,$$

we can further rewrite expression (13) as

$$E \;=\; \frac{1}{8\pi G} \int d^3x\, \partial_i (e e^{ai} e^{bj}\, {}^o\omega_{jab}) \;. \tag{15}$$

Up to this point $\{e_{a\mu}\}$ are tetrads of the four-dimensional space-time. In order to compare (15) with (1) let us impose the time gauge $e^{(0)}{}_k = e_{(j)}{}^0 = 0$ and establish the 3+1 decomposition of the tetrads as in [7, 22]. Then the integrand on the right hand side of (15) can be rewritten as

$$^4e\, {}^4e^{ai}\, {}^4e^{bj}\, {}^o\omega_{jab}({}^4e) = N e e^{(m)i} e^{(n)j}\, {}^o\omega_{j(m)(n)}(e) - e(e^{ai} N^j - e^{aj} N^i)\, {}^o\omega_{j(0)(m)}$$

N and N^i are the lapse and shift functions and $\{{}^4e^{a\mu}\}$ are tetrads of the four-dimensional space-time. In terms of triads restricted to a three-dimensional spacelike surface we have

$$E \;=\; \frac{1}{8\pi G} \int d^3x \partial_i [N e e^{(m)i} e^{(n)j}\, {}^o\omega_{j(m)(n)} - e(e^{(m)i} N^j - e^{(m)j} N^i) {}^o\omega_{j(0)(m)}] \;. \tag{16a}$$

Comparison with (1) can now be made if we make use of the *identity*[23]

$$\partial_i (e e^{(m)i} e^{(n)j}\, {}^o\omega_{j(m)(n)}) \equiv \partial_i (e T^i) \;,$$

where the right hand side above is the same as in (1). Møller energy can be finally written as

$$E \;=\; \frac{1}{8\pi G} \int d^3x \partial_i [N e T^i - e(e^{(m)i} N^j - e^{(m)j} N^i)\, {}^o\omega_{j(0)(m)})] \;. \tag{16b}$$

Recall that we are ignoring *local* Lorentz transformations.

Besides the appearance of extra terms involving ${}^o\omega_{j(0)(m)}$ on the right hand side of (16a,b), there is also the crucial presence of the lapse function N multiplying eT^i. Therefore even for configurations of the gravitational field for which the second term on the right hand side of (16) does not contribute (if, say, $N^i = 0$, as for the Schwarzschild solution) expressions (1) and (16) will lead to different results when applied to finite volumes of the three-dimensional space. Moreover, because of the presence of the lapse function, (16) is not invariant under time reparametrizations: $N'(x'^0) = \frac{\partial x'^0}{\partial x^0} N(x^0)$. Thus for a finite volume of integration (16b) does not remain invariant under this reparametrization.

In the Einstein-Cartan theory the connection ${}^o\omega_{j(0)(m)}$ can be expressed in terms of the momenta canonically conjugated to $e_{(m)i}$. In the notation of [22] it is given by ${}^o\omega_{j(0)(m)} = \frac{1}{2e}(\pi_{(m)j} - \frac{1}{2} e_{(m)j}\pi)$ (see equation 12 of [22]). In this context (16b) can be rewritten as

$$E \;=\; \frac{1}{8\pi G} \int d^3x \partial_i [NeT^i - \frac{1}{2} e^{(m)i} N^j \pi_{(m)j}] \;.$$

The expression above is exactly the energy expression for the Einstein-Cartan theory (see equation 21 of [22]), assuming that the gravitational energy is obtained from integration of surface terms of the total Hamiltonian. This expression is also very similar to (i) the integral of the surface terms in equation 6, and (ii) the energy expression considered by Nester[24] in the analysis of the positivity of the gravitational energy (equation (3.15) of [24]). All definitions of gravitational energy considered above agree for the total gravitational energy.

IV. Bondi's radiating metric and the associated triads

Bondi's metric is a not an exact solution of Einstein's equations. It describes the asymptotic form of a radiating solution. In terms of radiation coordinates (u, r, θ, ϕ), where u is the retarded time and r is the luminosity distance, Bondi's radiating metric is written as

$$ds^2 \;=\; -\Big(\frac{V}{r} e^{2\beta} - U^2\, r^2 e^{2\gamma}\Big) du^2 - 2e^{2\beta} du\, dr - 2U\, r^2\, e^{2\gamma} du\, d\theta$$

$$+r^2\Big(e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\,sin^2\theta\,d\phi^2\Big)\,. \tag{17}$$

The metric above is such that the surfaces for which $u = constant$ are null hypersurfaces. Each null radial (light) ray is labelled by particular values of u, θ and ϕ. At spacelike infinity u takes the standard form $u = t - r$. The four quantities appearing in (17), V, U, β and γ are functions of u, r and θ. Thus (17) displays axial symmetry. A more general form of this metric has been given by Sachs[25], who showed that the most general metric tensor describing asymptotically flat gravitational waves depends on six functions of the coordinates.

The functions in (17) satisfy the following asymptotic behaviour:

$$\beta \;=\; -\frac{c^2}{4r^2} + ...$$

$$\gamma \;=\; \frac{c}{r} + O(\frac{1}{r^3}) + ...$$

$$\frac{V}{r} \;=\; 1 - \frac{2M}{r} - \frac{1}{r^2}\Big[\frac{\partial d}{\partial\theta} + d\,cot\theta - \Big(\frac{\partial c}{\partial\theta}\Big)^2 - 4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta - \frac{1}{2}\,c^2\Big(1+8cot^2\theta\Big)\Big] + ...$$

$$U \;=\; \frac{1}{r^2}\Big(\frac{\partial c}{\partial\theta} + 2c\,cot\theta\Big) + \frac{1}{r^3}\Big(2d + 3c\,\frac{\partial c}{\partial\theta} + 4c^2\,cot\theta\Big) + ...$$

where $M = M(u,\theta)$ and $d = d(u,\theta)$ are the mass aspect and the dipole aspect, respectively, and from the function $c(u,\theta)$ we define the news function $\frac{\partial c(u,\theta)}{\partial u}$.

The application of (1) to Bondi's metric requires transforming it to coordinates t, r, θ and ϕ for which $t = constant$ defines a space-like hypersurface. Before proceeding, we recall that the analysis of (17) in t, x, y, z coordinates has already been performed by Goldberg[26], in the investigation of the asymptotic invariants of gravitational radiation fields. Therefore we carry out a coordinate transformation such that the new timelike coordinate is given by $t = u + r$. We arrive at

$$ds^2 \;=\; -\Big(\frac{V}{r}e^{2\beta} - U^2\,r^2\,e^{2\gamma}\Big)dt^2 - 2U\,r^2\,e^{2\gamma}dt\,d\theta + 2\Big[e^{2\beta}\Big(\frac{V}{r} - 1\Big) - U^2\,r^2\,e^{2\gamma}\Big]dr\,dt$$

$$+\left[e^{2\beta}\left(2-\frac{V}{r}\right)+U^2\,r^2\,e^{2\gamma}\right]dr^2+2U\,r^2\,e^{2\gamma}dr\,d\theta+r^2\left(e^{2\gamma}d\theta^2+e^{-2\gamma}\,sin^2\theta\,d\phi^2\right)\,. \tag{18}$$

Therefore the metric restricted to a three-dimensional spacelike hypersurface is given by

$$\begin{aligned} ds^2 \;&=\; \left[e^{2\beta}\left(2-\frac{V}{r}\right)+U^2\,r^2\,e^{2\gamma}\right]dr^2+2U\,r^2\,e^{2\gamma}\,dr\,d\theta \\ &\quad +r^2\left(e^{2\gamma}\,d\theta^2+e^{-2\gamma}\,sin^2\theta\,d\phi^2\right)\,. \end{aligned} \tag{19}$$

We must consider triads that correspond to the metric above. The construction of triads, in general, is a nontrivial step. If in a given coordinate system the metric tensor is diagonal, then the construction of triads is a relatively simple procedure. One must only make sure that the triads satisfy the appropriate boundary conditions at infinity. Recall that in order to obtain expression (9) for the ADM energy the triads must have the appropriate asymptotic behaviour given by equation (8).

The metric tensor (19) has an off-diagonal element, and this fact adds a bit of complication in the construction of triads. Nevertheless we can immediately write down two sets of triads that lead to this metric. They are given by

$$e_{(k)i} \;=\; \begin{pmatrix} A\,sin\theta\,cos\phi+B\,cos\theta\,cos\phi & rC\,cos\theta\,cos\phi & -rD\,sin\theta\,sin\phi \\ A\,sin\theta\,sin\phi+B\,cos\theta\,sin\phi & rC\,cos\theta\,sin\phi & rD\,sin\theta\,cos\phi \\ A\,cos\theta-B\,sin\theta & -rC\,sin\theta & 0 \end{pmatrix}\,, \tag{20}$$

where

$$A \;=\; e^{\beta}\sqrt{2-\frac{V}{r}}\,, \tag{21a}$$

$$B \;=\; r\,U\,e^{\gamma}\,, \tag{21b}$$

$$C \;=\; e^{\gamma}\,, \tag{21c}$$

$$D = e^{-\gamma} , \tag{21d}$$

and

$$e_{(k)i} = \begin{pmatrix} A'\, sin\theta\, cos\phi & rB'\, cos\theta\, cos\phi + rC'\, sin\theta\, cos\phi & -rD'\, sin\theta\, sin\phi \\ A'\, sin\theta\, sin\phi & rB'\, cos\theta\, sin\phi + rC'\, sin\theta\, sin\phi & rD'\, sin\theta\, cos\phi \\ A'\, cos\theta & -rB'\, sin\theta + rC'\, cos\theta & 0 \end{pmatrix} , \tag{22}$$

where

$$A' = \left[e^{2\beta} \left(2 - \frac{V}{r} \right) + U^2\, r^2\, e^{2\gamma} \right]^{\frac{1}{2}} , \tag{23a}$$

$$B' = \frac{1}{A'}\, e^{\beta+\gamma} \sqrt{2 - \frac{V}{r}} , \tag{23b}$$

$$C' = \frac{1}{A'}\, U\, r\, e^{2\gamma} , \tag{23c}$$

$$D' = e^{-\gamma} . \tag{23d}$$

It is easy to see that both (20) and (22) yield the metric tensor (19) through the relation $e_{(i)j} e_{(i)k} = g_{jk}$. They are related by a *local* SO(3) transformation.

Triads given by (20) and (22) are the *simplest* sets of triads that satisfy the two basic requirements: (**i**) the triads must have the asymptotic behaviour given by (8); (**ii**) by making the physical parameters of the metric vanish we must have $T_{(k)ij} = 0$ everywhere. In the present case if we make $M = d = c = 0$ both (20) and (22) acquire the form

$$e_{(k)i} = \begin{pmatrix} sin\theta\, cos\phi & r\, cos\theta\, cos\phi & -r\, sin\theta\, sin\phi \\ sin\theta\, sin\phi & r\, cos\theta\, sin\phi & r\, sin\theta\, cos\phi \\ cos\theta & -r\, sin\theta & 0 \end{pmatrix} . \tag{24}$$

In cartesian coordinates the expression above can be reduced to the diagonal form $e_{(k)i}(x, y, z) = \delta_{ik}$. The requirement (ii) above is essentialy equivalent to the establishment of a reference space, as discussed in [9]. Without the notion of a reference space we cannot define gravitational energy. Note that

by a suitable choice of a local SO(3) rotation we can make the flat space triad (24) satisfy the requirement (i), but not (ii).

It is possible to show that (20) and (22) are particular cases of an infinit set of triads that satisfy both requirements above. It is given by

$$e_{(k)i} = \begin{pmatrix} \mathcal{A}\, sin\theta cos\phi + \mathcal{B}\, cos\theta cos\phi & r\mathcal{C}\, cos\theta cos\phi + r\mathcal{D}\, sin\theta cos\phi & -re^{-\gamma}\, sin\theta sin\phi \\ \mathcal{A}\, sin\theta sin\phi + \mathcal{B}\, cos\theta sin\phi & r\mathcal{C}\, cos\theta sin\phi + r\mathcal{D}\, sin\theta sin\phi & re^{-\gamma}\, sin\theta cos\phi \\ \mathcal{A}\, cos\theta - \mathcal{B}\, sin\theta & -r\mathcal{C}\, sin\theta + r\mathcal{D}\, cos\theta & 0 \end{pmatrix} , \tag{25}$$

with the following definitions:

$$\mathcal{A} = \sqrt{e^{2\beta}\left(2 - \frac{V}{r}\right) + U^2 r^2 \left(e^{2\gamma} - \mathbf{b}^2\right)} ,$$

$$\mathcal{B} = \mathbf{b}\, Ur ,$$

$$\mathcal{C} = \sqrt{e^{2\gamma} - \mathbf{d}^2\, U^2 r^2} ,$$

$$\mathcal{D} = \mathbf{d}\, Ur ,$$

where $\mathbf{b}$ and $\mathbf{d}$ are arbitrary, dimensionless functions that must satisfy

$$\mathbf{b} = \sqrt{e^{2\gamma} - \mathbf{d}^2\, U^2 r^2} + \mathbf{d}\, e^{-\gamma+\beta}\sqrt{2 - \frac{V}{r}}$$

By making $\mathbf{d} = 0$ we obtain (20), and $\mathbf{b} = 0$ leads to (22).

From the point of view of the TEGR triads given by (20) and (22) are physically inequivalent (that is, they are not gauge equivalent), because we have seen that the Hamiltonian formulation established by eq.(5) is not invariant under the local SO(3) group, but rather under the global SO(3). In the TEGR the torsion tensor describes the way in which the space-time is deformed. The latter is thus considered as a continuum with microstructure[5]. Therefore the same space, defined uniquely by the metric tensor, may be deformed in several ways, according to the manner one defines the triads. This is essentially the geometrical meaning of the noninvariance of the TEGR under local SO(3) transformations.

In the Hamiltonian formulation of the TEGR the basic geometrical field variable is the triad, not the metric tensor. Any set of triads should be ruled out on physical grounds, i.e., if they lead to incorrect physical statements concerning the energy content of the gravitational field.

In the next section we will obtain the expressions for the gravitational energy arising from (20) and (22). These expressions are quite different. Although the expression corresponding to (20) is simpler, as we will see, we have no definite experimental evidence in favour of it.

V. Gravitational radiation energy

In this section we will apply expression (1) both to (20) and (22). Since the two triads display distinct geometrical properties, we expect to obtain different expressions for the energy density $\frac{1}{8\pi}\partial_i(eT^i)$ (we will make the gravitational constant $G=1$). Our analysis is meaningful only in the asymptotic region of large values of the radial distance. However, we have no reason to expect (20) and (22) to yield different expressions for the *total* energy of the field. In fact, as we will see, they yield the same (expected) expression.

As we mentioned earlier, the significance of the present approach to the analysis of gravitational radiation fields resides in the fact that we can evaluate the gravitational energy inside a large but finite portion of a three-dimensional spacelike surface. In other words, by means of Gauss law (1) can be evaluated over a surface far from the source of radiation, in the asymptotic limit where the metric components are precisely determined. Specifically, we will evaluate (1) inside a large sphere of radius r_o. The time evolution of the metric field will determine the time dependence of this energy, and consequently the energy radiated out of it.

In order to calculate $T^i = g^{ik}g^{mj}e_{(l)m}T_{(l)jk}$ we need the inverse metric g^{ij}. In terms of the definitions (21), it is given by

$$g^{ij} \;=\; \begin{pmatrix} \frac{1}{A^2} & -\frac{B}{r\,e^{\gamma}\,A^2} & 0 \\ -\frac{B}{r\,e^{\gamma}\,A^2} & \frac{1}{r^2\,e^{2\gamma}}(1+\frac{B^2}{A^2}) & 0 \\ 0 & 0 & \frac{1}{r^2\,e^{-2\gamma}\,sin^2\theta} \end{pmatrix}\,. \tag{26}$$

We will initially consider the set of triads given by (20). In the following, a comma after a field quantity indicates a derivative: $A_{,1}$ and $A_{,2}$ indicate

derivative with of A respect to r and θ, respectively. The torsion components for (20) are given by

$$T_{(1)12} = cos\theta\, cos\phi\, (C + rC,_1 - A - B,_2\,) + sin\theta\, cos\phi\, (-A,_2 + B)\ ,$$

$$T_{(1)13} = sin\theta\, sin\phi\, (A - D - rD,_1\,) + cos\theta\, sin\phi\, B\ ,$$

$$T_{(1)23} = sin\theta\, sin\phi\, (-rD,_2\,) + cos\theta\, sin\phi\, (rC - rD)\ ,$$

$$T_{(2)12} = cos\theta\, sin\phi\, (C + rC,_1 - A - B,_2\,) + sin\theta\, sin\phi\, (-A,_2 + B)\ ,$$

$$T_{(2)13} = sin\theta\, cos\phi\, (-A + D + rD,_1\,) + cos\theta\, cos\phi\, (-B)\ ,$$

$$T_{(2)23} = sin\theta\, cos\phi\, (rD,_2\,) + cos\theta\, cos\phi\, (-rC + rD)\ ,$$

$$T_{(3)12} = sin\theta\, (A + B,_2 - C - rC,_1\,) + cos\theta\, (-A,_2 + B)\ ,$$

$$T_{(3)13} = T_{(3)23} = 0\ .$$

Since we are interested in calculating the energy inside a surface of constant radius, only T^1 will be considered. By Gauss law, the expression of this energy is given by

$$E\ =\ \frac{1}{8\pi}\int_S d\theta\, d\phi\, eT^1\ , \qquad (27)$$

where S is a surface of fixed radius r_o, assumed to be large as compared with the dimension of the source, and the determinant e is given by $e = r^2\, A\, sin\theta$. After a long but otherwise straightforward calculation we arrive at

$$E_I\ =\ \frac{r_o}{4}\int_0^\pi d\theta\left\{sin\theta\Big[e^\gamma + e^{-\gamma} - \frac{2}{A}\Big] + \frac{1}{A}\frac{\partial}{\partial\theta}(Ur\, sin\theta)\right\}\ . \qquad (28)$$

with A defined by (21a). Let us note that the field quantities appearing in (28) are functions of $u = t-r$: $M = M(t-r,\theta), c = c(t-r,\theta)$, etc. Therefore,

once these functions are known, one can explicitely calculate the variation of E_I with respect to the time t, however only in the limit where the metric components are precisely determined.

Unfortunately the expansion of E_I up to terms in $\frac{1}{r_o}$, making use of the aymptotic behaviour of U, V, β and γ, yields no simple expression. It is given by

$$E_I = \frac{1}{2}\int_0^\pi d\theta\, sin\theta\, M - \frac{1}{4r_o}\int_0^\pi d\theta\, sin\theta\Big[\Big(\frac{\partial c}{\partial\theta}\Big)^2 + 4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta + 4c^2\, cot^2\theta\Big]$$

$$-\frac{1}{4r_o}\int_0^\pi d\theta\, M\,\frac{\partial}{\partial\theta}\Big[sin\theta\Big(\frac{\partial c}{\partial\theta} + 2c\, cot\theta\Big)\Big]\ . \tag{29}$$

In the calculation above we have assumed that $U(\theta)sin\theta = d(\theta)sin\theta = 0$ when $\theta = 0$ or π.

We observe that E_I yields Bondi energy in the limit $r \to \infty$ in the *static* case (i.e., when M is a function of θ only), so that E_I is the total gravitational energy. However, in the nonstatic case an expression for the loss of mass due to gravitational radiation can be obtained from (29). This is one major result of our analysis.

Let us recall that Bondi's *mass aspect* $m(u)$,

$$m(u) = \frac{1}{2}\int_0^\pi d\theta\, sin\theta\, M(u,\theta)\ , \tag{30}$$

depends on the null foliation used. The mass aspect $m(u)$ can be understood as a mass associated to each null cone determined by the equation $u = constant$. Since the limit $r \to \infty$ corresponds to $u \to -\infty$ for finite t, we see once again that in this limit E_I gives the total energy because it corresponds to the initial value of the Bondi energy.

We will consider next the second set of triads, eq.(22). The components of the torsion tensor resulting from the latter are given by

$$T_{(1)12} = cos\theta\, cos\phi\,(B' + rB'_{,1} - A') + sin\theta\, cos\phi\,(C' + rC'_{,1} - A'_{,2})\ ,$$

$$T_{(1)13} = sin\theta\, sin\phi\,(A' - D' - rD'_{,1})\ ,$$

$$T_{(1)23} = cos\theta\, sin\phi\, (rB' - rD') + sin\theta\, sin\phi\, (rC' - rD'{,_2}) \ ,$$

$$T_{(2)12} = cos\theta\, sin\phi\, (B' + rB'{,_1} - A') + sin\theta\, sin\phi\, (C' + rC'{,_1} - A'{,_2}) \ ,$$

$$T_{(2)13} = -sin\theta\, cos\phi\, (A' - D' - rD'{,_1}) \ ,$$

$$T_{(2)23} = -cos\theta\, cos\phi\, (rB' - rD') - sin\theta\, cos\phi\, (rC' - rD'{,_2}) \ ,$$

$$T_{(3)12} = -sin\theta\, (B' + rB'{,_1} - A') + cos\theta\, (C' + rC'{,_1} - A'{,_2}) \ ,$$

$$T_{(3)13} \ = \ T_{(3)23} \ = 0 \ .$$

As in the previous case, we are interested in calculating the energy in the interior of a surface of constant radius r_o. Therefore only the knowledge of T^1 will be necessary. After a long calculation we first arrive at

$$\begin{aligned} eT^1 \ &= \ -\frac{r\, sin\theta}{A}\Big\{e^{-2\gamma}\Big[B'\frac{\partial}{\partial r}(rB') + C'\frac{\partial}{\partial r}(rC') - A'B' - C'\frac{\partial A'}{\partial \theta}\Big] \\ &\qquad + e^{2\gamma}\Big[-A'e^{-\gamma} - r\frac{\partial \gamma}{\partial r}e^{-2\gamma} + \ e^{-2\gamma}\Big]\Big\} \\ &\qquad -\frac{rB\, sin\theta}{A}\Big[C' + \frac{\partial\gamma}{\partial\theta}\, e^{-\gamma}\Big] - \frac{rB\, cos\theta}{A}\Big[B' - e^{-\gamma}\Big] \ , \end{aligned} \tag{31}$$

where the primed quantities are given by (23). It is not straightforward to put the expression above in a simplified form. After some rearrangements we can finally write the energy expression (27) as

$$\begin{aligned} E_{II} \ &= \ \frac{r_o}{4}\int_0^\pi d\theta\, \frac{1}{A}\Big\{sin\theta\Big[e^{\gamma}A' + e^{-2\gamma}A'B' - 2 + e^{-2\gamma}\frac{\partial A'}{\partial\theta}C' - BC' - Be^{-\gamma}\frac{\partial\gamma}{\partial\theta}\Big] \\ &\qquad -B\, cos\theta\Big[B' - e^{-\gamma}\Big]\Big\} \ . \end{aligned} \tag{32}$$

Like equation (28), this expression represents the energy enclosed by a large spherical surface of radius r_o. Expanding the expression above up to the first power of $\frac{1}{r_o}$ we find

$$E_{II} \;=\; \frac{1}{2}\int_0^\pi d\theta\, M\, sin\theta - \frac{1}{4r_o}\int_0^\pi d\theta\, sin\theta\Big[3M^2 + \frac{5}{2}\Big(\frac{\partial c}{\partial\theta}\Big)^2 + 10c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta$$

$$+8c^2cot^2\theta - \Big(\frac{\partial M}{\partial\theta}\Big)\Big(\frac{\partial c}{\partial\theta} + 2c\,cot\theta\Big)\Big] - \frac{1}{4r_o}\int_0^\pi d\theta\, cos\theta\Big[2c\Big(\frac{\partial c}{\partial\theta}\Big) + 4c^2\, cot\theta\Big]\,. \tag{33}$$

We are again assuming $U(\theta)sin\theta = d(\theta)sin\theta = 0$ for $\theta = 0, \pi$.

We observe that in the limit $r \to \infty$ $\quad E_{II}$ also gives the total energy. As before, for a finite (but sufficiently large) value of r_o we can compute the loss of mass due to gravitational radiation, once the functions M and c are known in the asymptotic region.

VI. The selection of triads

In section IV we obtained an infinit set of triads that yield the three-dimensional spacelike section of Bondi's metric, and in the previous section we considered in detail only the simplest constructions. Of course simplicity is a major feature of physical systems, but we are really in need of experimental evidence that leads to a definite description. We need actual realizations of the quantities $M(r-t,\theta)$, $c(r-t,\theta)$, $d(r-t,\theta)$ and experimental evidence on how the energy is radiated away in order to arrive simultaneously at the correct energy expression arising from (27) and at the definite expression of $e_{(k)i}$.

However, we can envisage two possible types of conditions on the triads that associate a unique triad with the three-dimensional metric tensor.

The first condition regards the energy content of the gravitational field. If we stick to the point of view according to which physical systems in nature have a tendency to be in states of minimum energy, then the correct triad for the spacelike section of Bondi's metric is the one that minimizes expression (27) for all possible constructions of $e_{(k)i}$. By means of this criterium we

consider triads given by (25), or any further construction that complies with the two conditions stated in section IV, and ask which one yields the smaller value of energy contained within a surface of constant radius, in similarity with the calculations of (29) and (33). Unfortunately this analysis cannot be carried out unless M, c and d are known.

Certainly one can ask whether only (27) should be minimized or the energy density should be everywhere a minimum. In the context of Bondi's metric the latter possibility cannot be considered, because the metric is valid only in the asymptotic limit, but in the general case it is an open question that must be carefully addressed.

The second condition takes into account equation (8): we require the triads to have the asymptotic behaviour determined by (8) with a *symmetric* tensor $h_{jk} = h_{kj}$. Again, one has to find out of (25) which realization of $e_{(i)k}$ in cartesian coordinates complies with this criterium. This condition may be understood as a *rotational gauge condition*. Note that as it stands, h_{jk} in equation (8) is not required to be symmetric (in equation (9) only the symmetrical part contributes). By explicit calculations we observe that neither (20) nor (22) satisfy this second condition.

The two conditions above may not be mutually excluding. On the contrary, they may lead to the same triad. The determination of the correct triad is certainly an essential and crucial issue of the theory and will be further investigated elsewhere in the general case, with special attention to Bondi's metric, in the light of the conditions above.

We observed that both (29) and (33) yield the same total energy. This is also the case if we carry out the calculations with a more complicated triad, whether belonging to (25) or not, which is related to (20) or (22) by a local SO(3) transformation with an appropriate asymptotic behaviour. Let us consider a local SO(3) transformation given by

$$\tilde{e}^{(k)}{}_i(x) \;=\; \Lambda^{(k)}{}_{(l)}(x)\, e^{(l)}{}_i(x) \;. \qquad (34)$$

Under (34) the energy expression (1) transforms as

$$\tilde{E} \;=\; E + \frac{1}{8\pi}\int_V d^3x \partial_i [e g^{ik} \Lambda^{(l)}{}_{(m)} e^{(m)j}\, (e_{(n)k}\partial_j \Lambda_{(l)}{}^{(n)} - e_{(n)j}\partial_k \Lambda_{(l)}{}^{(n)})] \;. \quad (35)$$

Expression (35) can be best analysed if we consider an infinitesimal rotation.

We assume that in the limit $r \to \infty$ the SO(3) elements have the asymptotic behaviour

$$\Lambda^{(k)}{}_{(l)} \approx \delta^{(k)}_{(l)} + {}^0\omega^{(k)}{}_{(l)} + {}^1\omega^{(k)}{}_{(l)}(\frac{1}{r}) \ ,$$

such that ${}^{0,1}\omega_{(k)(l)} = -{}^{0,1}\omega_{(l)(k)}$, and $\{{}^0\omega_{(k)(l)}\}$ are constants. Taking into account (8) it is easy to see that when integrated over the whole three-dimensional space the integral on the right hand side of (35) reduces to a vanishing expression:

$$\frac{1}{8\pi}\int_{V\to\infty} d^3x(\partial_i\partial_j\,{}^1\omega_{(i)(j)} - \partial_i\partial_i\,{}^1\omega_{(j)(j)}) = 0 \ .$$

Therefore we expect to find the same result for the total energy if we evaluate (27) out of any element of the set of triads (25).

VII. Discussion

The application of the energy definition (1) for a given solution of Einstein's equations requires considering a foliation of the space-time in three-dimensional spacelike surfaces. The metric for the spacelike section of Bondi's radiating metric admits an infinit set of triads related by local SO(3) transformations. In the present case, from this infinit set of triads we singled out two of them. We have considered in detail the two ones that exhibit the simplest structures in spherical coordinates, and that: (i) satisfy the asymptotic conditions given by (8); (ii) reduce to the reference space ($T_{(k)ij} = 0$ everywhere) if we make the physical parameters vanish: $M = c = d = 0$.

The two sets of triads, (20) and (22), describe the spacelike section of Bondi's metric given by (19) and lead to the energy expressions (28) and (32), respectively. These expressions establish distinct and quite definite physical predictions. They allow us to compute the energy radiated from the interior of a spherical surface of constant radius r_o. It would be a remarkable achievement of the TEGR if, on physical grounds, we could decide for one of them or even for an arbitrary element of (25). In the TEGR the space (space-time) geometry is fundamentally described by triads (tetrads). Unfortunately we do not dispose of experimental information for taking such a decision.

It is a very important result that the *total* energy due to both sets of triads (as well as from any element of (25)) agrees exactly with the static Bondi

energy, in which case the energy arises from the integration of $M = M(\theta)$. In fact, the definition of Bondi's *mass aspect* is basically motivated by the fact that in the static case, and by investigating the asymptotic properties of the gravitational field, $M(\theta)$ arises as the mass of an isolated system.

The final expressions (29) and (33) support the consistency of the definition (1), and the relevance of the TEGR as a fundamental description of general relativity. However the present analysis, which was developed on spacelike surfaces, has to be compared with the one recently carried out on null surfaces[27]. Let us recall that in order to obtain the Hamiltonian formulation given by (5-7) we imposed the time gauge condition. Therefore the resulting geometry may be understood as a *three-dimensional* teleparallel geometry, since the teleparallelism is restricted to the three-dimensional spacelike surface. On the other hand, in the Hamiltonian formulation developed in [27] we have not fixed any particular tetrad component, and consequently the teleparallel geometry is really *four-dimensional*.

In [27] the constraints also contain a total divergence, in similarity with (7a), and may be taken likewise to define the gravitational energy-momentum. Although it appears that the geometrical framework of [27] is better suited to the analysis of the Bondi-Sachs metric, we note that the the energy expression arising there is considerably more complicated than (1). Moreover we do not know yet whether the constraint algebra leads to a consistent Hamiltonian formulation (either on null or spacelike surfaces). We also note that one Hamiltonian formulation cannot be reduced to the other by means of gauge fixing. Nature admits only one correct physical description, and therefore either the three-dimensional or the four-dimensional teleparallel geometry is the correct candidate for describing the energy properties of the gravitational field. All these issues will be considered in the near future.

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